

## CONSTRUCTION OF AN EIGHT DIMENSIONAL NONALTERNATIVE, NONCOMMUTATIVE ALGEBRA

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ABSTRACT. The purpose of this article is to construct a unital 8 dimensional hypercomplex number system  $H_8^*$  that is neither alternative nor commutative unlike the octonions by means of the unital 4 dimensional, commutative, and nonassociative hypercomplex number system  $H^*$ . We also establish some algebraic properties related to  $H_8^*$  and compare to those of octonions.

### 1. Introduction

Number systems  $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O} \subset \dots$  can be extended from the previous number system. But,  $\mathbb{H} \subset \mathbb{O} \subset \dots$  doesn't guarantee the commutative property. In [5], we introduced the commutative number systems.

Properties of quaternions and matrices of quaternions was published [6], but the noncommutativity was a big obstacle to expand the theory. In 1892, Segre introduced commutative quaternions [3] and various properties about commutative quaternions have been established [1,2,3]. Also, we introduced modified 4 dimensional commutative quaternions  $H^*$  which is not associative [4].

As is well known, octonions  $\mathbb{O}$  was constructed by means of Cayley-Dickson construction with quaternions  $\mathbb{H}$ . Note that  $\mathbb{O}$  is a unital 8 dimensional, nonalternative, noncommutative, division algebra.

Recall the 4 dimensional, nonassociative, commutative quaternions  $H^*$ .

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DEFINITION 1.1. ([4]) Let

$H^* = \{a_0 + a_1i + a_2j + a_3k \mid a_0, a_1, a_2, a_3 \in \mathbb{R}, i, j, k \notin \mathbb{R}\}$ , where

$$i^2 = k^2 = -1, j^2 = 1, ij = ji = k, jk = kj = -i, ki = ik = j.$$

Then,  $H^*$  is a unital 4 dimensional, nonassociative, commutative  $\mathbb{Z}_2$ -graded algebra. Moreover,  $H^*$  is neither an alternative algebra nor a division algebra.

In this paper, we will construct a unital 8 dimensional, nonalternative, noncommutative hypercomplex number system  $H_8^*$  by means of the algebra  $H^*$ .

## 2. Eight dimensional nonalternative, noncommutative algebra

In this section, we will construct a unital 8 dimensional nonalternative, noncommutative, nondivision algebra by using the 4 dimensional nonalternative, commutative, nondivision algebra  $H^*$ .

Let  $\alpha = a_0 + a_1i + a_2j + a_3k \in H^*$  for some  $a_0, a_1, a_2, a_3 \in \mathbb{R}$ . Instead of defining the conjugate of  $\alpha \in H^*$  by  $a_0 - a_1i - a_2j - a_3k$ , we simply define the conjugate  $\alpha^{(1)}$  of  $\alpha$  as follows:

DEFINITION 2.1. Let  $\alpha = a_0 + a_1i + a_2j + a_3k \in H^*$  for some  $a_0, a_1, a_2, a_3 \in \mathbb{R}$ . Then, we define  $\alpha^{(1)} = a_0 - a_1i - a_2j + a_3k$  as the conjugate of  $\alpha$ .

Note that  $\alpha = \alpha_1 + \alpha_2k$  for some  $\alpha_1, \alpha_2 \in \mathbb{C}$  and

$$\alpha^{(1)} = (a_0 - a_1i) + (a_3 - a_2i)k = \overline{\alpha_1} + \overline{\alpha_2}k,$$

where  $\overline{\alpha_t}$  is the complex conjugate of  $\alpha_t$ ,  $t = 1, 2$ .

The conjugate is an involution satisfying the following facts.

PROPOSITION 2.2. Let  $\alpha, \beta \in H^*$ . Then,

- (1)  $\alpha^{(1)} = \alpha$  if and only if  $\alpha \in \mathbb{R}$ .
- (2)  $(\alpha^{(1)})^{(1)} = \alpha$ .
- (3)  $(\alpha + \beta)^{(1)} = \alpha^{(1)} + \beta^{(1)}$ .
- (4)  $(a\alpha)^{(1)} = a\alpha^{(1)}$  for all  $a \in \mathbb{R}$ .
- (5)  $(\alpha\beta)^{(1)} = \alpha^{(1)}\beta^{(1)}$ .

Starting with the unital 4 dimensional, nonassociative, commutative, nondivision  $\mathbb{Z}_2$ -graded algebra  $H^*$ , a unital 8 dimensional nonalternative, noncommutative, nondivision  $\mathbb{Z}_2$ -graded algebra  $H_8^*$  is constructed as follows:

DEFINITION 2.3. Let  $H_8^* = \{(\alpha, \beta) | \alpha, \beta \in H^*\}$  and define the operations on the set  $H_8^*$  as follows:

$$\begin{aligned} (\alpha, \beta) + (\gamma, \eta) &= (\alpha + \gamma, \beta + \eta) \\ a(\alpha, \beta) &= (a\alpha, a\beta) \\ (\alpha, \beta)(\gamma, \eta) &= (\alpha\gamma - \beta\eta^{(1)}, \alpha\eta + \beta\gamma^{(1)}) \end{aligned}$$

for all elements  $(\alpha, \beta), (\gamma, \eta) \in H_8^*$  and  $a \in \mathbb{R}$ .

PROPOSITION 2.4.  $H_8^*$  is a unital 8 dimensional nonalternative, noncommutative, and nondivision algebra.

*Proof.* Obviously, the set  $\mathcal{B} = \{(1, 0), (i, 0), (j, 0), (k, 0), (0, 1), (0, i), (0, j), (0, k)\}$  is a basis of  $H_8^*$ . Note that

$$\begin{aligned} \{(i, j)(i, j)\}(i, i) &= (ii - j(-j), ij + j(-i))(i, i) = (0, 0)(i, i) = (0, 0) \\ (i, j)\{(i, j)(i, i)\} &= (i, j)(ii - j(-i), ii + j(-i)) = (i, j)(-1 + k, -1 - k) \\ &= (i(-1 + k) - j(-1 - k), i(-1 - k) + j(-1 + k)) \\ &= 2(-i + j, -i - j) \end{aligned}$$

Thus,  $H_8^*$  is not alternative. Also,  $H_8^*$  is not commutative since

$$(0, 1)(0, i) = (i, 0) \neq (-i, 0) = (0, i)(0, 1)$$

To show  $H_8^*$  is not a division algebra, consider the element  $(1 + j, 0) \in H_8^*$ . Then,

$$(1 + j, 0)(\alpha, \beta) = ((1 + j)\alpha, (1 + j)\beta^{(1)}) = (1, 0)$$

implies that

$$(1 + j)\alpha = 1, \quad (1 + j)\beta^{(1)} = 0$$

If we let  $\alpha = a_0 + a_1i + a_2j + a_3k$  and  $\beta = b_0 + b_1i + b_2j + b_3k$ , then

$$a_0 + a_2 = 1, \quad a_1 - a_3 = 0, \quad a_0 + a_2 = 0, \quad a_1 + a_3 = 0,$$

which is impossible. Thus, the element  $(1 + j, 0)$  has no multiplicative inverse. Consequently,  $H_8^*$  is a unital 8 dimensional, nonalternative, noncommutative, and nondivision algebra.  $\square$

The multiplication table of basis members of the algebra  $H_8^*$  is as follows:

	(1, 0)	(i, 0)	(j, 0)	(k, 0)	(0, 1)	(0, i)	(0, j)	(0, k)
(1, 0)	(1, 0)	(i, 0)	(j, 0)	(k, 0)	(0, 1)	(0, i)	(0, j)	(0, k)
(i, 0)	(i, 0)	(-1, 0)	(k, 0)	(j, 0)	(0, i)	(0, -1)	(0, k)	(0, j)
(j, 0)	(j, 0)	(k, 0)	(1, 0)	(-i, 0)	(0, j)	(0, k)	(0, 1)	(0, -i)
(k, 0)	(k, 0)	(j, 0)	(-i, 0)	(-1, 0)	(0, k)	(0, j)	(0, -i)	(0, -1)
(0, 1)	(0, 1)	(0, -i)	(0, -j)	(0, k)	(-1, 0)	(i, 0)	(j, 0)	(-k, 0)
(0, i)	(0, i)	(0, 1)	(0, -k)	(0, j)	(-i, 0)	(-1, 0)	(k, 0)	(-j, 0)
(0, j)	(0, j)	(0, -k)	(0, -1)	(0, -i)	(-j, 0)	(k, 0)	(1, 0)	(i, 0)
(0, k)	(0, k)	(0, -j)	(0, i)	(0, -1)	(-k, 0)	(j, 0)	(-i, 0)	(1, 0)

Let  $e_1 = (1, 0)$ ,  $e_2 = (i, 0)$ ,  $e_3 = (j, 0)$ ,  $e_4 = (k, 0)$ ,  $e_5 = (0, 1)$ ,  $e_6 = (0, i)$ ,  $e_7 = (0, j)$ ,  $e_8 = (0, k)$ . Then we have

	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$	$e_7$	$e_8$
$e_1$	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$	$e_7$	$e_8$
$e_2$	$e_2$	$-e_1$	$e_4$	$e_3$	$e_6$	$-e_5$	$e_8$	$e_7$
$e_3$	$e_3$	$e_4$	$e_1$	$-e_2$	$e_7$	$e_8$	$e_5$	$-e_6$
$e_4$	$e_4$	$e_3$	$-e_2$	$-e_1$	$e_8$	$e_7$	$-e_6$	$-e_5$
$e_5$	$e_5$	$-e_6$	$-e_7$	$e_8$	$-e_1$	$e_2$	$e_3$	$-e_4$
$e_6$	$e_6$	$e_5$	$-e_8$	$e_7$	$-e_2$	$-e_1$	$e_4$	$-e_3$
$e_7$	$e_7$	$-e_8$	$-e_5$	$-e_6$	$-e_3$	$e_4$	$e_1$	$e_2$
$e_8$	$e_8$	$-e_7$	$e_6$	$-e_5$	$-e_4$	$e_3$	$-e_2$	$e_1$

THEOREM 2.5. Let  $R_0$  and  $R_1$  be two sets defined by

$$R_0 = \left\{ \sum_{p,q,r,s=1}^4 ae_p e_q e_r e_s \mid a \in \mathbb{R} \right\}, \quad R_1 = \left\{ \sum_{p,q,r,s=5}^8 ae_p e_q e_r e_s \mid a \in \mathbb{R} \right\}.$$

Then,  $R_0 \cong H^*$  and  $R_1 = H_8^*$ . Moreover,  $H_8^*$  is a  $\mathbb{Z}_2$ -graded algebra.

The  $\mathbb{Z}_2$ -graded algebra  $H_8^*$  is not alternative, but the subset

$$V = \left\{ \sum_{p=5}^8 a_p e_p \mid a_p \in \mathbb{R}, p = 5, 6, 7, 8 \right\}$$

satisfies similar property.

THEOREM 2.6. Let  $V = \left\{ \sum_{p=5}^8 a_p e_p \mid a_p \in \mathbb{R}, p = 5, 6, 7, 8 \right\}$ . Then,

$$(e_p e_p) e_q = e_p (e_p e_q) \quad \text{or} \quad (e_p e_p) e_q = -e_p (e_p e_q)$$

for all  $e_p, e_q \in R_1, 5 \leq p, q \leq 8$ .

*Proof.* The proof is straightforward.

$$(e_5 e_5) e_6 = -e_6 = e_5 (e_5 e_6), \quad (e_5 e_5) e_7 = -e_7 = e_5 (e_5 e_7), \quad (e_5 e_5) e_8 = -e_8 = e_5 (e_5 e_8)$$

$$(e_6 e_6) e_5 = -e_5 = e_6 (e_6 e_5), \quad (e_6 e_6) e_7 = -e_7, \quad e_6 (e_6 e_7) = e_7, \quad (e_6 e_6) e_8 = -e_8, \quad e_6 (e_6 e_8) = e_8$$

$$(e_7 e_7) e_5 = e_5 = e_7 (e_7 e_5), \quad (e_7 e_7) e_6 = e_6, \quad e_7 (e_7 e_6) = -e_6, \quad (e_7 e_7) e_8 = e_8, \quad e_7 (e_7 e_8) = -e_8,$$

$$(e_8 e_8) e_5 = e_5 = e_8 (e_8 e_5), \quad (e_8 e_8) e_6 = e_6 = e_8 (e_8 e_6), \quad (e_8 e_8) e_7 = e_7 = e_8 (e_8 e_7)$$

□

Now, we will construct a real matrix representation of elements in  $H_8^*$ .

LEMMA 2.7. Let  $\alpha, \beta \in H^*$  and let  $\alpha = a_0 + a_1 i + a_2 j + a_3 k, \beta = b_0 + b_1 i + b_2 j + b_3 k$ . Then,

$$\begin{aligned} (\alpha, \beta)(1, 0) &= (\alpha, \beta) = (a_0 + a_1 i + a_2 j + a_3 k, b_0 + b_1 i + b_2 j + b_3 k) \\ (\alpha, \beta)(i, 0) &= (\alpha i, \beta i^{(1)}) = (\alpha i, -\beta i) \\ &= (-a_1 + a_0 i + a_3 j + a_2 k, b_1 - b_0 i - b_3 j - b_2 k) \\ (\alpha, \beta)(j, 0) &= (\alpha j, \beta j^{(1)}) = (\alpha j, -\beta j) \\ &= (a_2 - a_3 i + a_0 j + a_1 k, -b_2 + b_3 i - b_0 j - b_1 k) \\ (\alpha, \beta)(k, 0) &= (\alpha k, \beta k^{(1)}) = (\alpha k, \beta k) \\ &= (-a_3 - a_2 i + a_1 j + a_0 k, -b_3 - b_2 i + b_1 j + b_0 k) \\ (\alpha, \beta)(0, 1) &= (-\beta, \alpha) = (-b_0 - b_1 i - b_2 j - b_3 k, a_0 + a_1 i + a_2 j + a_3 k) \\ (\alpha, \beta)(0, i) &= (-\beta i^{(1)}, \alpha i) = (\beta i, \alpha i) \\ &= (-b_1 + b_0 i + b_3 j + b_2 k, -a_1 + a_0 i + a_3 j + a_2 k) \\ (\alpha, \beta)(0, j) &= (-\beta j^{(1)}, \alpha j) = (\beta j, \alpha j) \\ &= (b_2 - b_3 i + b_0 j + b_1 k, a_2 - a_3 i + a_0 j + a_1 k) \\ (\alpha, \beta)(0, k) &= (-\beta k^{(1)}, \alpha k) = (-\beta k, \alpha k) \\ &= (b_3 + b_2 i - b_1 j - b_0 k, -a_3 - a_2 i + a_1 j + a_0 k) \end{aligned}$$

Define the map  $\phi^{(1)} : H_8^* \longrightarrow M_{8 \times 8}(\mathbb{R})$  by

$$\phi^{(1)}(\alpha, \beta) = \begin{pmatrix} a_0 & -a_1 & a_2 & -a_3 & -b_0 & -b_1 & b_2 & b_3 \\ a_1 & a_0 & -a_3 & -a_2 & -b_1 & b_0 & -b_3 & b_2 \\ a_2 & a_3 & a_0 & a_1 & -b_2 & b_3 & b_0 & -b_1 \\ a_3 & a_2 & a_1 & a_0 & -b_3 & b_2 & b_1 & -b_0 \\ b_0 & b_1 & -b_2 & -b_3 & a_0 & -a_1 & a_2 & -a_3 \\ b_1 & -b_0 & b_3 & -b_2 & a_1 & a_0 & -a_3 & -a_2 \\ b_2 & -b_3 & -b_0 & b_1 & a_2 & a_3 & a_0 & a_1 \\ b_3 & -b_2 & -b_1 & b_0 & a_3 & a_2 & a_1 & a_0 \end{pmatrix}$$

for all elements  $(\alpha, \beta) \in H_8^*$ . Then,

$$\phi^{(1)}(\alpha, \beta) = \begin{pmatrix} \phi_1(\alpha) & -\phi_1(\beta)G_1 \\ \phi_1(\beta)G_1 & \phi_1(\alpha) \end{pmatrix},$$

where

$$\phi_1(\alpha) = \begin{pmatrix} a_0 & -a_1 & a_2 & -a_3 \\ a_1 & a_0 & -a_3 & -a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_3 & a_2 & a_1 & a_0 \end{pmatrix} \quad \text{and} \quad G_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

**THEOREM 2.8.** Let  $\alpha_t, \beta_t \in H^*$ ,  $t = 1, 2$  and  $a \in \mathbb{R}$ . Then,

- (1)  $\phi^{(1)}(a(\alpha_1, \beta_1)) = a\phi^{(1)}(\alpha_1, \beta_1)$ .
- (2)  $\phi^{(1)}(\alpha_1 + \alpha_2, \beta_1 + \beta_2) = \phi^{(1)}(\alpha_1, \beta_1) + \phi^{(1)}(\alpha_2, \beta_2)$ .
- (3)  $\phi^{(1)}((\alpha_1, \beta_1)(\alpha_2, \beta_2)) \neq \phi^{(1)}(\alpha_1, \beta_1)\phi^{(1)}(\alpha_2, \beta_2)$  in general.

*Proof.* (1) and (2) are obvious.

For (3),  $\phi^{(1)}((0, 1)(0, 1)) = \phi^{(1)}((-1, 0)) = -I_8 \neq \phi^{(1)}(0, 1)\phi^{(1)}(0, 1)$ .  $\square$

**THEOREM 2.9.** Let  $\alpha, \beta \in H^*$  and let  $\alpha = a_0 + a_1i + a_2j + a_3k$ ,  $\beta = b_0 + b_1i + b_2j + b_3k$ . Then,

- (1) If  $\phi_1(\alpha)$  is invertible if and only if  $\phi^{(1)}(\alpha, 0)$  is invertible.
- (2) If  $\phi_1(\beta)$  is invertible if and only if  $\phi^{(1)}(0, \beta)$  is invertible.
- (3) If  $\phi^{(1)}(\alpha, \beta)$  is invertible and  $\phi_1(\alpha) = 0$ , then  $\phi_1(\beta)$  is invertible.
- (4) If  $\phi^{(1)}(\alpha, \beta)$  is invertible and  $\phi_1(\beta) = 0$ , then  $\phi_1(\alpha)$  is invertible.
- (5)  $\text{tr}(\phi^{(1)}(\alpha, \beta)) = 8a_0 = 8\text{Re}(\alpha) = 2\text{tr}(\phi_1(\alpha))$ .

*Proof.* (1) If  $\phi_1(\alpha)$  is invertible, then

$$\phi^{(1)}(\alpha, 0) \begin{pmatrix} \phi_1(\alpha)^{-1} & O \\ O & \phi_1(\alpha)^{-1} \end{pmatrix} = I_8 = \begin{pmatrix} \phi_1(\alpha)^{-1} & O \\ O & \phi_1(\alpha)^{-1} \end{pmatrix} \phi^{(1)}(\alpha, 0)$$

and thus  $\phi^{(1)}(\alpha, 0)$  is invertible.

Conversely, if  $\phi^{(1)}(\alpha, 0)$  is invertible, then

$$\begin{pmatrix} \phi_1(\alpha) & O \\ O & \phi_1(\alpha) \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = I_8 = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} \phi_1(\alpha) & O \\ O & \phi_1(\alpha) \end{pmatrix}$$

for some  $C_{11}, C_{12}, C_{21}, C_{22} \in M_{4 \times 4}(\mathbb{R})$ . Thus,  $\phi_1(\alpha)C_{11} = I_4$  and  $\phi_1(\alpha)$  is invertible.

(2) If  $\phi_1(\beta)$  is invertible, then

$$\phi^{(1)}(0, \beta) \begin{pmatrix} O & G_1^{-1}\phi_1(\beta)^{-1} \\ -G_1^{-1}\phi_1(\beta)^{-1} & O \end{pmatrix} = I_8$$

and thus  $\phi^{(1)}(0, \beta)$  is invertible.

Conversely, if  $\phi^{(1)}(0, \beta)$  is invertible, then

$$\begin{pmatrix} O & -\phi_1(\beta)G_1 \\ \phi_1(\beta)G_1 & O \end{pmatrix} \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} = I_8$$

for some  $D_{11}, D_{12}, D_{21}, D_{22} \in M_{4 \times 4}(\mathbb{R})$ . Thus,  $\phi_1(\beta)G_1D_{12} = I_4$  and  $\phi_1(\beta)$  is invertible.

(3) If  $\phi^{(1)}(\alpha, \beta)$  is invertible and  $\phi_1(\alpha) = 0$ , then  $\phi^{(1)}(0, \beta)$  is invertible. Thus, by (2),  $\phi_1(\beta)$  is invertible.

(4) If  $\phi^{(1)}(\alpha, \beta)$  is invertible and  $\phi_1(\beta) = 0$ , then  $\phi^{(1)}(\alpha, 0)$  is invertible. Thus, by (1),  $\phi_1(\alpha)$  is invertible.

(5) is obvious by the definition of  $\phi^{(1)}(\alpha, \beta)$ . □

### 3. Matrices in $M_{n \times n}(H_8^*)$

Let  $\psi^{(1)} : M_{n \times n}(H_8^*) \longrightarrow M_{8n \times 8n}(\mathbb{R})$  be the map defined by  $\psi^{(1)}(A) = C$ , where  $A = (A_{st})_{n \times n}$ ,  $A_{st} = (\alpha_{st}, \beta_{st})$ ,  $C = (C_{st})_{8n \times 8n}$ , and

$$C_{st} = \phi^{(1)}(A_{st}) = \begin{pmatrix} \phi_1(\alpha_{st}) & -\phi_1(\beta_{st})G_1 \\ \phi_1(\beta_{st})G_1 & \phi_1(\alpha_{st}) \end{pmatrix}$$

the  $(s, t)$  - th  $8 \times 8$  block matrix.

DEFINITION 3.1. Let  $A, B \in M_{n \times n}(H_8^*)$ . Then, the  $8n \times 8n$  matrix  $\psi^{(1)}(A)$  is called the adjoint matrix of  $A$ .

DEFINITION 3.2. Let  $A, B \in M_{n \times n}(H_8^*)$ . Then, we define the determinant of the matrix  $A$  by the determinant of  $\psi^{(1)}(A)$ .

THEOREM 3.3. Let  $A, B \in M_{n \times n}(H_8^*)$ . Then,

- (1)  $\psi^{(1)}(aA) = a\psi^{(1)}(A)$  for all  $a \in \mathbb{R}$ .
- (2)  $\psi^{(1)}(A + B) = \psi^{(1)}(A) + \psi^{(1)}(B)$ .
- (3)  $\psi^{(1)}(AB) \neq \psi^{(1)}(A)\psi^{(1)}(B)$  in general.
- (4)  $\det(aA) = a^{8n} \det(A)$ .
- (5)  $\det(AB) \neq \det(A) \det(B)$  for  $n \geq 2$  in general.

*Proof.* Let  $A = (A_{st})_{n \times n}$ ,  $A_{st} = (\alpha_{st}, \beta_{st})$ ,  $B = (B_{st})_{n \times n}$ ,  $B_{st} = (\gamma_{st}, \delta_{st})$  for some  $\alpha_{st}, \beta_{st}, \gamma_{st}, \delta_{st} \in H_8^*$ . Then,

(1) The  $(s, t)$  - th  $8 \times 8$  block matrix of  $\psi^{(1)}(aA)$  is

$$\begin{aligned} \phi^{(1)}(aA_{st}) &= \phi^{(1)}(a(\alpha_{st}, \beta_{st})) = \phi^{(1)}(a\alpha_{st}, a\beta_{st}) = a\phi^{(1)}(\alpha_{st}, \beta_{st}) \\ &= a\phi^{(1)}(A_{st}) \end{aligned}$$

Since the  $(s, t)$  - th  $8 \times 8$  block matrix of  $a\psi^{(1)}(A)$  is  $a\phi^{(1)}(A_{st})$ , we have  $\psi^{(1)}(aA) = a\psi^{(1)}(A)$ .

(2) The  $(s, t)$  - th  $8 \times 8$  block matrix of  $\psi^{(1)}(A + B)$  is

$$\begin{aligned} \phi^{(1)}(A_{st} + B_{st}) &= \phi^{(1)}(\alpha_{st} + \gamma_{st}, \beta_{st} + \delta_{st}) \\ &= \begin{pmatrix} \phi_1(\alpha_{st} + \gamma_{st}) & -\phi_1(\beta_{st} + \delta_{st})G_1 \\ \phi_1(\beta_{st} + \delta_{st})G_1 & \phi_1(\alpha_{st} + \gamma_{st}) \end{pmatrix} \\ &= \begin{pmatrix} \phi_1(\alpha_{st}) & -\phi_1(\beta_{st})G_1 \\ \phi_1(\beta_{st})G_1 & \phi_1(\alpha_{st}) \end{pmatrix} + \begin{pmatrix} \phi_1(\gamma_{st}) & -\phi_1(\delta_{st})G_1 \\ \phi_1(\delta_{st})G_1 & \phi_1(\gamma_{st}) \end{pmatrix} \\ &= \phi^{(1)}(\alpha_{st}, \beta_{st}) + \phi^{(1)}(\gamma_{st}, \delta_{st}) = \phi^{(1)}(A_{st}) + \phi^{(1)}(B_{st}) \end{aligned}$$

which is the  $(s, t)$  - th  $8 \times 8$  block matrix of  $\psi^{(1)}(A) + \psi^{(1)}(B)$ .

(3) Let  $A = (i, 0)I_n$  and  $B = (j, 0)I_n$ . Then,  $AB = (k, 0)I_n$  and

$$\begin{aligned} \psi^{(1)}(A) &= \psi^{(1)}((i, 0)I_n) = \psi^{(1)}(\text{diag}((i, 0), (i, 0), \dots, (i, 0))) \\ &= \text{diag} \left( \begin{pmatrix} \phi_1(i) & -\phi_1(0)G_1 \\ \phi_1(0)G_1 & \phi_1(i) \end{pmatrix}, \dots, \begin{pmatrix} \phi_1(i) & -\phi_1(0)G_1 \\ \phi_1(0)G_1 & \phi_1(i) \end{pmatrix} \right) \\ &= \text{diag} \left( \begin{pmatrix} \phi_1(i) & O \\ O & \phi_1(i) \end{pmatrix}, \dots, \begin{pmatrix} \phi_1(i) & O \\ O & \phi_1(i) \end{pmatrix} \right) \end{aligned}$$



$$\begin{aligned}
 \psi^{(1)}(B) &= \psi^{(1)}((j, 0)I_n) = \psi^{(1)}(\text{diag}((j, 0), (j, 0), \dots, (j, 0))) \\
 &= \text{diag} \left( \left( \begin{array}{cc} \phi_1(j) & -\phi_1(0)G_1 \\ \phi_1(0)G_1 & \phi_1(j) \end{array} \right), \dots, \left( \begin{array}{cc} \phi_1(j) & -\phi_1(0)G_1 \\ \phi_1(0)G_1 & \phi_1(j) \end{array} \right) \right) \\
 &= \text{diag} \left( \left( \begin{array}{cc} \phi_1(j) & O \\ O & \phi_1(j) \end{array} \right), \dots, \left( \begin{array}{cc} \phi_1(j) & O \\ O & \phi_1(j) \end{array} \right) \right)
 \end{aligned}$$

$$\begin{aligned}
 \psi^{(1)}(AB) &= \psi^{(1)}((k, 0)I_n) = \psi^{(1)}(\text{diag}((k, 0), (k, 0), \dots, (k, 0))) \\
 &= \text{diag} \left( \left( \begin{array}{cc} \phi_1(k) & -\phi_1(0)G_1 \\ \phi_1(0)G_1 & \phi_1(k) \end{array} \right), \dots, \left( \begin{array}{cc} \phi_1(k) & -\phi_1(0)G_1 \\ \phi_1(0)G_1 & \phi_1(k) \end{array} \right) \right) \\
 &= \text{diag} \left( \left( \begin{array}{cc} \phi_1(k) & O \\ O & \phi_1(k) \end{array} \right), \dots, \left( \begin{array}{cc} \phi_1(k) & O \\ O & \phi_1(k) \end{array} \right) \right)
 \end{aligned}$$

Since  $\phi_1(i)\phi_1(j) \neq \phi_1(k)$ , we have  $\psi^{(1)}(AB) \neq \psi^{(1)}(A)\psi^{(1)}(B)$ .

$$\begin{aligned}
 (4) \det(aA) &= \det(\psi^{(1)}(aA)) = \det(a\psi^{(1)}(A)) = a^{8n} \det(\psi^{(1)}(A)) \\
 &= a^{8n} \det(A).
 \end{aligned}$$

(5) Let  $A = (A_{st})_{n \times n}$ ,  $A_{st} = (0, 1)$ ,  $C = (C_{st})_{8n \times 8n}$ . Then,

$$C_{st} = \phi^{(1)}(A_{st}) = \phi^{(1)}(0, 1) = \begin{pmatrix} \phi_1(0) & -\phi_1(1)G_1 \\ \phi_1(1)G_1 & \phi_1(0) \end{pmatrix} = \begin{pmatrix} O & -G_1 \\ G_1 & O \end{pmatrix}$$

and so  $\det(A) = \det(\psi^{(1)}(A)) = \det(C) = 0$ .

Also,  $AA = nB$ , where  $B = (B_{st})_{n \times n}$ ,  $B_{st} = (1, 0)$ . Hence

$$\det(AA) = \det(\psi^{(1)}(AA)) = \det(\psi^{(1)}(nB)) = n^{8n} \det(I_{8n}) = n^{8n}.$$

Thus,  $\det(AA) \neq \det(A)\det(A)$ .  $\square$

Note that every element  $\alpha \in H^*$  can be uniquely expressed as  $\alpha = \alpha_1 + \alpha_2 k$  for some  $\alpha_1, \alpha_2 \in \mathbb{C}$ . Thus, every element  $(\alpha, \beta) \in H_8^*$  can be uniquely expressed as  $(\alpha, \beta) = (\alpha_1 + \alpha_2 k, \beta_1 + \beta_2 k)$  for some  $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$ . Let

$$G = \{(\alpha_1 + \alpha_2 k, \beta_1 + \beta_2 k) \in H_8^* \mid \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}\}$$

and define the map  $\theta : G \rightarrow G$  by  $\theta(\gamma, \eta) = (\alpha, \beta)(\gamma, \eta)$  for all  $(\gamma, \eta) \in G$ . Then,

$$\begin{aligned}
 \theta(1, 0) &= \alpha_1(1, 0) + \alpha_2(k, 0) + \beta_1(0, 1) + \beta_2(0, k), \\
 \theta(k, 0) &= (-\alpha_2)(1, 0) + \alpha_1(k, 0) + (-\beta_2)(0, 1) + \beta_1(0, k), \\
 \theta(0, 1) &= (-\beta_1)(1, 0) + (-\beta_2)(k, 0) + \alpha_1(0, 1) + \alpha_2(0, k), \\
 \theta(0, k) &= \beta_2(1, 0) + (-\beta_1)(k, 0) + (-\alpha_2)(0, 1) + \alpha_1(0, k)
 \end{aligned}$$

Thus, we establish the function  $\phi_{\mathbb{C}}^{(1)} : G \longrightarrow M_{4 \times 4}(\mathbb{C})$  defined by

$$\phi_{\mathbb{C}}^{(1)}((\alpha, \beta)) = \begin{pmatrix} \alpha_1 & -\alpha_2 & -\beta_1 & \beta_2 \\ \alpha_2 & \alpha_1 & -\beta_2 & -\beta_1 \\ \beta_1 & -\beta_2 & \alpha_1 & -\alpha_2 \\ \beta_2 & \beta_1 & \alpha_2 & \alpha_1 \end{pmatrix}.$$

**THEOREM 3.4.** *Let  $(\alpha, \beta), (\gamma, \eta) \in G$ . Then the followings are satisfied:*

- (1)  $\phi_{\mathbb{C}}^{(1)}(a(\alpha, \beta)) = a\phi_{\mathbb{C}}^{(1)}(\alpha, \beta)$  for all  $a \in \mathbb{R}$ .
- (2)  $\phi_{\mathbb{C}}^{(1)}((\alpha, \beta) + (\gamma, \eta)) = \phi_{\mathbb{C}}^{(1)}(\alpha, \beta) + \phi_{\mathbb{C}}^{(1)}(\gamma, \eta)$ .
- (3)  $\phi_{\mathbb{C}}^{(1)}((\alpha, \beta)(\gamma, \eta)) \neq \phi_{\mathbb{C}}^{(1)}(\alpha, \beta)\phi_{\mathbb{C}}^{(1)}(\gamma, \eta)$  in general.

*Proof.* (1) and (2) are straightforward and we shall prove (3). Let  $\alpha = \alpha_1 + \alpha_2k = i$  and  $\gamma = \gamma_1 + \gamma_2k = j$  for some  $\alpha_t, \gamma_t \in \mathbb{C}$ ,  $t = 1, 2$ . Then  $\alpha_1 = i$ ,  $\alpha_2 = 0$ ,  $\gamma_1 = 0$ ,  $\gamma_2 = i$ . Thus,

$$\begin{aligned} \phi_{\mathbb{C}}^{(1)}((\alpha, 0)(\gamma, 0)) &= \phi_{\mathbb{C}}^{(1)}((k, 0)) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ &\neq \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \\ &= \phi_{\mathbb{C}}^{(1)}((\alpha, 0))\phi_{\mathbb{C}}^{(1)}((\gamma, 0)) \end{aligned}$$

and so  $\phi_{\mathbb{C}}^{(1)}((\alpha, \beta)(\gamma, \eta)) \neq \phi_{\mathbb{C}}^{(1)}((\alpha, \beta))\phi_{\mathbb{C}}^{(1)}((\gamma, \eta))$  in general. □

Let  $\mathbb{C}^2 = \{(\alpha, \beta) \mid \alpha \in \mathbb{C}\}$ . Then,  $A \in M_{n \times n}(H_{\mathbb{8}}^*)$  is uniquely expressed by  $A = A_1 + A_2k$  for some  $A_1, A_2 \in M_{n \times n}(\mathbb{C}^2)$ .

**DEFINITION 3.5.** Let  $A \in M_{n \times n}(H_{\mathbb{8}}^*)$ . Then,  $\lambda \in H_{\mathbb{8}}^*$  is a left eigenvalue of  $A$  if  $AX = \lambda X$  for some  $X \neq O \in M_{n \times 1}(H_{\mathbb{8}}^*)$ .

**THEOREM 3.6.** *Let  $A = A_1 + A_2k \in M_{n \times n}(H_{\mathbb{8}}^*)$  for some  $A_1, A_2 \in M_{n \times n}(\mathbb{C}^2)$  and  $\lambda = \lambda_1 + \lambda_2k$  for some  $\lambda_1, \lambda_2 \in \mathbb{C}^2$ . If  $\lambda$  is a left eigenvalue of  $A$  if and only if there exists a nonzero matrix  $X = X_1 + X_2k \in M_{n \times 1}(H_{\mathbb{8}}^*)$  for some  $X_1, X_2 \in M_{n \times 1}(\mathbb{C}^2)$  such that*

$$\begin{pmatrix} A_1 - \lambda_1 I_n & -A_2 + \lambda_2 I_n \\ A_2 - \lambda_2 I_n & A_1 - \lambda_1 I_n \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} O \\ O \end{pmatrix}.$$

*Proof.* Note that

$$\begin{aligned}(A_2k)(X_2k) &= -A_2X_2, (\lambda_2k)(X_2k) = -\lambda_2X_2, \\ A_1(X_2k) + (A_2k)X_1 &= (A_1X_2 + A_2X_1)k, \\ \lambda_1(X_2k) + (\lambda_2k)X_1 &= (\lambda_1X_2 + \lambda_2X_1)k.\end{aligned}$$

Thus,  $AX = \lambda X$  is equivalent to

$$A_1X_1 - A_2X_2 = \lambda_1X_1 - \lambda_2X_2, \quad A_1X_1 - A_2X_2 = \lambda_1X_1 - \lambda_2X_2$$

which is equivalent to  $\begin{pmatrix} A_1 - \lambda_1 I_n & -A_2 + \lambda_2 I_n \\ A_2 - \lambda_2 I_n & A_1 - \lambda_1 I_n \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} O \\ O \end{pmatrix}$ .  $\square$

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